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COSET WEIGHT ENUMERATORS AND A GREEN'S FUNCTION OVER A HAMMING SCHEME WITH APPLICATION TO CHEEGER'S CONSTANTS

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Abstract. We show that the Hamming weight enumerators of cosets of a linear code $\mathcal{C}$ over $\mathbb{F}_q$ of length n are closely related to a discrete Green's function $\mathcal{G}_\beta$. Using this relation, for instance, we obtain the exact number of distinct coset weight enumerators of $\mathcal{C}$. As an application, we show that lower bounds of the Cheeger ratio and the Cheeger constant of $\mathcal{C}$ on Γ_1 can be explicitly determined by $\mathcal{G}_\beta$ and a coset weight enumerator value of $\mathcal{C}$, where Γ_1 is a distance regular graph with respect to a Hamming distance 1 over $\mathbb{F}_q^n$.

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1. Introduction

The concept of a Green's function was introduced in a famous essay of George Green in 1728. In [4], a discrete Green's function defined on graphs is closely associated with the normalized Laplacian and is useful for solving discrete Laplace equations with boundary conditions. In [2, 3], F. Chung introduced the relationship between the PageRank and a discrete Green's function $\mathcal{G}_\beta$ with a positive real number β. A Green's function $\mathcal{G}_\beta$ can be explained with an inverse relation of the β-normalized Laplacian $\mathcal{L}_\beta$ represented by a adjacency matrix.

In [9], Delsarte introduced a Hamming scheme $H(n, q)$. A Hamming scheme $H(n, q)$ is a P-polynomial scheme (metric scheme) [1, 9]. That is, for all i with $0 \leq i \leq n$, an adjacency matrix A_i with respect to the Hamming distance i can be represented by a polynomial of degree i with respect to an adjacency matrix A_1. Equivalently, a graph Γ_1 with respect to a Hamming distance 1 is a distance regular graph. In [7], an association scheme is constructed by the distinct coset weight enumerators of a linear code $\mathcal{C}$ over $\mathbb{F}_q^n$, and it turns out to be a P-polynomial scheme. It is also known that the coset weight enumerators of a linear code $\mathcal{C}$ are closely connected with the number of distinct nonzero weights of a dual code of $\mathcal{C}$ [6].

In this paper we show that the Hamming weight enumerators of cosets of a linear code $\mathcal{C}$ over $\mathbb{F}_q$ of length n are closely related to a discrete Green's function $\mathcal{G}_\beta$ by finding an explicit expression for $\mathcal{G}_\beta$ in terms of adjacency matrices (Theorem 2). Using this relation, for instance, we obtain the exact number of distinct coset weight enumerators of $\mathcal{C}$ (Theorem 6). In fact, we use an $n \times (n+1)$ matrix L_{sub} for the explicit expression of $\mathcal{G}_\beta$. As an application, we show that lower bounds of the Cheeger ratio and the

Cheeger constant of $\mathcal{C}$ on Γ_1 can be explicitly determined by β, $\alpha^{(0)}$ and r_0, where $\alpha^{(0)}$ is a coset weight enumerator value of $\mathcal{C}$ as in Definition 4, r_0 is given as in Theorem 2 and Γ_1 is a distance regular graph with respect to a Hamming distance 1 over $\mathbb{F}_q^n$. The Cheeger ratio and the Cheeger constant are related to the notion of PageRank by F. Chung [2].

This paper is organized as follows. In Section 2, we introduce some basic facts about the Hamming scheme $H(n,q)$ and a discrete Green's function $\mathcal{G}_\beta$, and in Section 3, we introduce an $n \times (n+1)$ matrix L_{sub}. In Section 4, we find an an explicit expression of a Green's function $\mathcal{G}_\beta$ over $\mathbb{F}_q^n$ in terms of adjacency matrices, and we explain the relationship between a Green's function $\mathcal{G}_\beta$ and L_{sub}. In Section 5, we show the relationship between a Green's function $\mathcal{G}_\beta$ and the coset weight enumerators of a linear code $\mathcal{C}$ over $\mathbb{F}_q$ of length n. Finally, in Section 6, we obtain the lower bounds of the Cheeger ratio and the Cheeger constant of $\mathcal{C}$ on Γ_1 in an explicit way by using the results in the sections 4 and 5.

2. Preliminaries

In this section, we introduce basic facts on Hamming scheme $H(n,q)$ and the discrete Green's function. Let $d_H(x,y)$ be the Hamming distance over $\mathbb{F}_q^n$. We describe the relations by their adjacency matrices A_i $(0 \le i \le n)$ is the $q^n \times q^n$ matrix defined by

$$(A_i)_{x,y} = \begin{cases} 1, & \text{if } d_H(x,y) = i \\ 0, & \text{otherwise.} \end{cases}$$

The Bose-Mesner algebra $\mathcal{A}$ generated by the adjacency matrices A_i , that is, $\mathcal{A} = \{\sum t_i A_i |\ t_0, t_1, \cdots, t_n \in \mathbb{R}\}$. Bose-Mesner algebra $\mathcal{A}$ has a unique basis of primitive idempotent matrices $E_0, E_1, \cdots, E_n$, that is,

$$(1)\ E_k E_l = \delta_{kl} E_k\ (k,l = 0,1,\cdots,n),\ (2)\ \sum_{i=0}^{n} E_i = I,$$

where δ_{kl} is the Kronecker delta. Bose-Mesner algebra $\mathcal{A}$ have two basis $\{A_i\}$ and $\{E_i\}$. We express the one in terms of the other and we obtain

$$A_j = \sum_{i=0}^{n} p_j(i) E_i,\ E_j = \frac{1}{q^n} \sum_{i=0}^{n} q_j(i) A_i \tag{1}$$

where $j = 0, 1, \cdots, n$. The $(n+1) \times (n+1)$ matrix $\mathbf{P} = (p_j(i))$(resp. $\mathbf{Q} = (q_j(i))$) is called the first eigenmatrix (resp. second eigenmatrix) of the Hamming scheme $H(n,q)$. They satisfy the relation $\mathbf{PQ} = \mathbf{QP} = q^n I$.

Define a transition probability matrix P by

$$P = \frac{1}{k_1} A_1,$$

where k_1 is the 1-th valency of $H(n,q)$. For a function $f : \mathbb{F}_q^n \to \mathbb{R}$, we define a *Laplace operator* Δ by

$$\Delta f(x) = \frac{1}{k_1} \sum_{y \sim x} (f(x) - f(y)).$$

Then, we have $\Delta = I - P$, i.e., $\Delta = I - \frac{1}{k_1}A_1$. For all $i = 0, 1, \cdots, n$, a Laplace operator Δ is symmetric since A_1 is symmetric. Then, Δ is a matrix representation of $\mathcal{L}$. For $j = 0, 1, \cdots, n$ and orthonormal eigenfunctions ϕ_j^*, we have

$$\mathcal{L} = \sum_{j=0}^{n} \lambda_j \phi_j^* \phi_j,$$

where λ_j is an eigenvalue of $\mathcal{L}$. Let $\mathcal{L}_\beta$ be the β-normalized Laplacian by $\beta I + \mathcal{L}$. For $\beta > 0$, let Green's function $\mathcal{G}_\beta$ denote the symmetric matrix satisfying $\mathcal{L}_\beta \mathcal{G}_\beta = I$. Then we have

$$\mathcal{G}_\beta = \sum_{j=0}^{n} \frac{1}{\beta + \lambda_j} \phi_j^* \phi_j.$$

For $\beta > 0$, we have

$$\mathcal{G}_\beta(\beta I + I - P) = I,$$

i.e.,

$$\mathcal{G}_\beta = ((\beta + 1)I - P)^{-1}.$$

Thus, by (1) this implies that

$$\begin{aligned}\mathcal{L}_\beta &= (\beta + 1)I - P = (\beta + 1)I - \frac{1}{k_1}A_1 \\ &= (\beta + 1)I - \frac{1}{k_1}\sum_{j=0}^{n} p_1(j)E_j.\end{aligned}$$

Since $I = E_0 + E_1 + \cdots + E_n$, we have

$$\begin{aligned}\mathcal{L}_\beta &= (\beta + 1)(E_0 + E_1 + \cdots + E_n) - \frac{1}{k_1}\left(p_1(0)E_0 + p_1(1)E_1 + \cdots + p_1(n)E_n\right) \\ &= \sum_{j=0}^{n} \left(\beta + 1 - \frac{1}{k_1}p_1(j)\right) E_j,\end{aligned}$$

where $\beta + 1 - \frac{1}{k_1}p_1(j)$ is an eigenvalue of $\mathcal{L}_\beta$. Hence, a Green's function $\mathcal{G}_\beta$ can be expressed by

$$\mathcal{G}_\beta = \sum_{j=0}^{n} \left(\frac{k_1}{(\beta + 1)k_1 - p_1(j)}\right) E_j.$$

Since $E_j = (1/q^n)\sum q_j(i)A_i$, the discrete Green's function $\mathcal{G}_\beta$ is a linear combination of adjacency matrices A_i.

The follows are the notation set used in this paper.

- $H(n, q)$: a Hamming scheme over $\mathbb{F}_q^n$.
- $\mathbf{P}$: the first eigenmatrix of $H(n, q)$.
- $\mathbf{Q}$: the second eigenmatrix of $H(n, q)$.
- $p_j(i)$: (i, j)-component of $\mathbf{P}$.
- $q_j(i)$: (i, j)-component of $\mathbf{Q}$.
- k_i : the valency of the i-th Hamming weight relation.
- $\mathcal{N}(A)$: a nullspace of a matrix A.
- $W(\mathcal{C})$: a weight enumerator of a set $\mathcal{C}$ over $\mathbb{F}_q$ of length n.
- $\mathbf{1}$: $q^n \times 1$ vector consisting of only '1'.
- Γ_1 : a graph with respect to a Hamming distance 1 over $\mathbb{F}_q^n$.

3. A reduction matrix L_{sub} on $\mathcal{L}_\beta$

Now, we introduce a $n \times (n+1)$ matrix L_{sub} obtained from $\mathcal{L}_\beta$. In fact, L_{sub} has information about the discrete Green's function $\mathcal{G}_\beta$. L_{sub} obtained by Step1, Step2, Step3 and Step4 as the following.

Let L be a $(q^n - 1) \times q^n$ matrix obtained by the removal of the first row of $\mathcal{L}_\beta$. Then we have the rank of L is $q^n - 1$, and the nullity is 1 since $\mathcal{G}_\beta$ has the inverse matrix. A basis of the nullspace of L can be induced from r_k's which are coefficients of A_k in $\mathcal{G}_\beta$. Let $\mathcal{G}_\beta^{(1)}$ be the first column vector of $\mathcal{G}_\beta$ which is arranged in order $r_0, r_1, \cdots, r_n$. Then $\mathcal{G}_\beta^{(1)}$ is a $q^n \times 1$ matrix, and we have

$$L\mathcal{G}_\beta^{(1)} = O$$

since $\mathcal{G}_\beta$ is orthogonal. Note that each row of $\mathcal{G}_\beta^{-1}$ is related to the Hamming weight of its row numbers. Thus, we can rearrange the rows of L by the Hamming weight of each row number as follow steps.

Step1 : Let

$$L = \begin{pmatrix} \mathbf{r}_{1\,1} \\ \vdots \\ \mathbf{r}_{1\,k_1} \\ \hline \mathbf{r}_{2\,1} \\ \vdots \\ \mathbf{r}_{2\,k_2} \\ \hline \vdots \\ \mathbf{r}_{n\,k_n} \end{pmatrix},$$

where $\mathbf{r}_{i\,k_i}$ is a row vector of L of Hamming weight i, and k_i is the number of row vectors of Hamming weight i.

Step2 : Let

$$L' = \begin{pmatrix} \mathbf{r}_{1\,1} \\ \mathbf{r}_{2\,1} \\ \vdots \\ \mathbf{r}_{n\,1} \end{pmatrix}.$$

Then, L' is an $n \times q^n$ matrix. Note that the column vectors of L is arranged according to its Hamming weight in the increasing weight order. We can rewrite L' with its column vectors as follows.

$$L' = \left(\begin{array}{c|ccc|ccc|cc} \mathbf{c}_{0\,1} & \mathbf{c}_{1\,1} & \cdots & \mathbf{c}_{1\,k_1} & \mathbf{c}_{2\,1} & \cdots & \mathbf{c}_{2\,k_2} & \cdots & \mathbf{c}_{n\,k_n} \end{array} \right),$$

where $\mathbf{c}_{i\,k_i}$ is a column vector of L' of Hamming weight i, and k_i is the number of column vectors of Hamming weight i.

Step3 : Let

$$L'' = \left(\begin{array}{ccccc} \mathbf{c}_{0\,1} & \sum_{i=1}^{k_1} \mathbf{c}_{1\,i} & \sum_{i=1}^{k_2} \mathbf{c}_{2\,i} & \cdots & \sum_{i=1}^{k_n} \mathbf{c}_{n\,i} \end{array} \right)$$

Then, L'' is an $n \times (n+1)$ matrix.

Step4 : Let $L_{sub} = -k_1 L''$ $(k_1 = n(q-1))$.

Now, we determine the entries of L_{sub}. Let $x, y \in \mathbb{F}_q^n$. For $x \in F_q^n$ with $wt_H(x) = i$, let s be the number of $y \in \mathbb{F}_q^n$ such that $wt_H(y) = j$ and

$d_H(x,y)=1$. Then we have :

$$s=\begin{cases} i, & j=i-1,\\ i(q-2), & j=i,\\ (n-i)(q-1), & j=i+1.\end{cases}$$

And, by the triangle inequality, $s=0$ if $|i-j|\geq 2$. Since L_{sub} is determined by Step 1,2,3,4 on $\mathcal{L}_\beta$, thus L_{sub} is as follows :

$$L_{sub}=\begin{pmatrix} 1 & s_1 & t_1 & 0 & 0 & \cdots & 0\\ 0 & 2 & s_2 & t_2 & 0 & \cdots & 0\\ 0 & 0 & 3 & s_3 & t_3 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0\\ 0 & 0 & 0 & \cdots & n-1 & s_{n-1} & t_{n-1}\\ 0 & 0 & 0 & \cdots & 0 & n & s_n \end{pmatrix},$$

where $s_i=i(q-2)-n(q-1)(\beta+1)$ for $i=1,2,\cdots,n$, and $t_j=(n-j)(q-1)$ for $j=1,2,\cdots,n-1$. It is clear that $s_{i-1}<s_i$ for $i=1,2,\cdots,n$ and $t_{j-1}>t_j>0$ for $j=1,2,\cdots,n-1$. L_{sub} is $n\times(n+1)$ matrix with $\text{rank}(L_{sub})=n$. Thus $\dim\mathcal{N}(L_{sub})=1$.

Example 1. For $\beta>0$, a 8×8 matrix $\mathcal{L}_\beta$ over $\mathbb{F}_2^3$ is as follows :

$$\mathcal{L}_\beta=\begin{pmatrix} \beta+1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0\\ -\frac{1}{3} & \beta+1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0\\ -\frac{1}{3} & 0 & \beta+1 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0\\ -\frac{1}{3} & 0 & 0 & \beta+1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0\\ 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & \beta+1 & 0 & 0 & -\frac{1}{3}\\ 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & \beta+1 & 0 & -\frac{1}{3}\\ 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & \beta+1 & -\frac{1}{3}\\ 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \beta+1 \end{pmatrix}.$$

Then, we obtain the following matrices L', L'' and L_{sub} respectively.

$$L'=\begin{pmatrix} -\frac{1}{3} & \beta+1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0\\ -\frac{1}{3} & 0 & \beta+1 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0\\ -\frac{1}{3} & 0 & 0 & \beta+1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0\\ 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & \beta+1 & 0 & 0 & -\frac{1}{3}\\ 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & \beta+1 & 0 & -\frac{1}{3}\\ 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & \beta+1 & -\frac{1}{3}\\ 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \beta+1 \end{pmatrix},$$

$$L''=\begin{pmatrix} -\frac{1}{3} & \beta+1 & -\frac{2}{3} & 0\\ 0 & -\frac{2}{3} & \beta+1 & -\frac{1}{3}\\ 0 & 0 & -\frac{3}{3} & \beta+1 \end{pmatrix},$$

$$L_{sub}=\begin{pmatrix} 1 & -3(\beta+1) & 2 & 0\\ 0 & 2 & -3(\beta+1) & 1\\ 0 & 0 & 3 & -3(\beta+1) \end{pmatrix}.$$

4. A discrete Green's function G_β and L_{sub}

In this section, we show that the Green's function $\mathcal{G}_\beta$ is determined by a basis of $\mathcal{N}(L_{sub})$. That is, the entries of $\mathcal{G}_\beta$ are determined by a basis $(u_0,u_1,\cdots,u_n)$ of $\mathcal{N}(L_{sub})$ with $u_n=1$.

In the following theorem, we find an explicit expression for a Green's function $\mathcal{G}_\beta$.

Theorem 2. *Let $\mathcal{G}_\beta$ be a Green's function over $\mathbb{F}_q^n$ and let $(u_0, u_1, \cdots, u_n)$ be a basis of $\mathcal{N}(L_{sub})$ with $u_n = 1$. Then we have the following.*

(i) A Green's function $\mathcal{G}_\beta$ can be expressed by a linear combination of adjacency matrices A_i of $\mathbb{F}_q^n$ for $i = 0, 1, 2, \cdots, n$ such as $\mathcal{G}_\beta = r_0A_0 + r_1A_1 + \cdots + r_nA_n$.

(ii) $(r_0, r_1, \cdots, r_n)$ is an element of $\mathcal{N}(L_{sub})$, and $(r_0, r_1, \cdots, r_n) = c(u_0, u_1, \cdots, u_n)$, where

$$c = (-1)^{n+1}\frac{n(q-1)n!}{tr\left[\prod_{i=0}^{n}\begin{pmatrix} s_{n-i} & -(n-i)t_{n-i-1} \\ 1 & 0 \end{pmatrix}\right]},\ t_{-1} = 0,$$

and $s_{n-i} = (n-i)(q-2) - n(q-1)(\beta+1), t_{n-i-1} = (i+1)(q-1)$.

(iii) r_i's satisfy that $r_0 > r_1 > \cdots > r_n > 0$.

Proof. (i) Let E_j be an idempotent matrix in $H(n, q)$. Then we have

$$E_j = \frac{1}{q^n}\sum_{i=0}^{n} q_j(i)A_i$$

so that, for $\beta > 0$,

$$\begin{aligned}
\mathcal{G}_\beta &= \sum_{j=0}^{n}\left(\frac{k_1}{(\beta+1)k_1 - p_1(j)}\right)E_j \\
&= \sum_{j=0}^{n}\left(\frac{k_1}{(\beta+1)k_1 - p_1(j)}\right)\frac{1}{q^n}\sum_{i=0}^{n} q_j(i)A_i \\
&= \frac{1}{q^n}\sum_{i=0}^{n}\sum_{j=0}^{n} q_j(i)\left(\frac{k_1}{(\beta+1)k_1 - p_1(j)}\right)A_i.
\end{aligned}$$

Hence, a Green's function $\mathcal{G}_\beta$ can be expressed by a linear combination of A_i.

(ii) Let $\mathcal{G}_\beta = r_0A_0 + r_1A_1 + \cdots + r_nA_n$. Then each element of the first row and the first column is related to the Hamming distance from the origin O in $\mathbb{F}_q^n$. we obtain that the first row of $\mathcal{G}_\beta$ is as follows :

$$\underbrace{r_0}_{1\ \text{term}}\ \underbrace{r_1\ \cdots\ r_1}_{k_1\ \text{terms}}\ \cdots\ \underbrace{r_n\ \cdots\ r_n}_{k_n\ \text{terms}}$$

Furthermore, $\mathcal{G}_\beta$ is a $q^n \times q^n$ matrix. Since $\mathcal{G}_\beta = ((\beta+1)I - P)^{-1}$, we have

$$\mathcal{G}_\beta^{-1} = (\beta+1)I - P = (\beta+1)I - \frac{1}{k_1}A_1$$

for $\beta > 0$. Since $(\beta+1)I - \frac{1}{k_1}A_1$ is symmetric, the first row(column) of $\mathcal{G}_\beta$ is orthogonal to the other rows(columns) of $(\beta+1)I - \frac{1}{k_1}A_1$. Since L is a matrix obtained from a removal of the first row of $\mathcal{G}_\beta^{-1}$, there is no row whose number is a Hamming weight 0. Hence, L_{sub} has n rows. Furthermore, each row of L_{sub} has $n+1$ elements. Therefore, we obtain an $n \times (n+1)$ matrix L_{sub}, and we have

$$L_{sub}\begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_n \end{pmatrix} = O_{n\times 1}.$$

Since $\mathcal{N}(L_{sub})$ is a 1-dimensional null space of L_{sub}, we have $(r_0, r_1, \cdots, r_n) = c(u_0, u_1, \cdots, u_n)$ for some real value c.

Now, we will determine a real value c. Let $\overline{L}_{sub}$ be an $(n+1)\times(n+1)$-matrix obtained from L_{sub} by adding $(s_0\ t_0\ 0\ \cdots\ 0)$ as its first row, where $s_0 = -n(q-1)(\beta+1)$,and $t_0 = n(q-1)$. Since $(r_0, r_1, \cdots, r_n)$ is orthogonal to the row vectors of L_{sub} ,and L_{sub} is obtained by multiplying $-n(q-1)$, the first column vector of $(-n(q-1))\overline{L}_{sub}^{-1}$ is $(r_0, r_1, \cdots, r_n)$. Therefore,

$$r_n = -n(q-1)\frac{\text{cofactor of }(1, n+1)\text{entry of } \overline{L}_{sub}}{\det(\overline{L}_{sub})},$$

where a cofactor of $(1, n+1)$-entry of $\overline{L}_{sub}$ is

$$(-1)^{2+n}\det\begin{pmatrix} 1 & s_1 & t_1 & \cdots & 0 \\ 0 & 2 & s_2 & t_2 & 0 \\ 0 & 0 & 3 & s_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & n \end{pmatrix} = (-1)^n\ n!.$$

Thus,

$$c = (-1)^{n+1}\frac{n(q-1)n!}{\det(\overline{L}_{sub})}.$$

Since $\overline{L}_{sub}$ is a tridiagonal matrix, $\det(\overline{L}_{sub})$ can be evaluated via multiplication of 2×2 matrices [8].

(iii) Now, for $k = 0, 1, 2\cdots, n$ and a coefficient r_k of A_k, we show that $r_0 > r_1 > \cdots > r_n > 0$. We claim that $i + t_i < -s_i$ for $i = 1, 2, \cdots, n-1$. since $n(q-1) - i(q-2) < n(q-1)(\beta+1) - i(q-2)$, we have

$$\begin{aligned} i + t_i &= i + (n-i)(q-1) \\ &= n(q-1) - i(q-2) \\ &< n(q-1)(\beta+1) - i(q-2) \\ &= -s_i. \end{aligned}$$

Hence, we have

$$i + t_i < -s_i \tag{2}$$

for $i = 1, 2, \cdots, n-1$.

We find a basis of the nullspace of L_{sub} by elementary row operations.

By the Gauss-Jordan elimination, we have the following reduced row echelon form matrix:

$$L_{sub} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & S_n \\ 0 & 1 & 0 & 0 & 0 & \cdots & S_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & S_2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & S_1 \end{pmatrix},$$

where $S_1 = \frac{s_n}{n}$, $S_2 = \frac{t_{n-1} - S_1 s_{n-1}}{n-1}$, and $S_i = -\frac{1}{n-i+1}\left(S_{i-1}s_{n-i+1} + S_{i-2}t_{n-i+1}\right)$ for $i = 3, 4, \cdots, n$. Then, since $\beta > 0$, we have

$$S_1 = \frac{s_n}{n} = q - 2 - (q-1)(\beta+1) < -1.$$

We claim that $S_i < S_{i-1}$ for $i = 1, 2, \cdots, n$. Note that

$$S_2 = \frac{1}{n-1}(t_{n-1} - S_1 s_{n-1}) = \frac{1}{n-1}(t_{n-1} + S_1(-s_{n-1}))$$

Since $S_1 < -1$ and $s_{n-1} < -1$, we have

$$\frac{1}{n-1}(t_{n-1} + S_1(-s_{n-1})) < \frac{1}{n-1}(t_{n-1} + S_1(n-1+t_{n-1}))$$

from (2). Since $S_1 < -1$, we have

$$S_2 < \frac{1}{n-1}(t_{n-1} + S_1(n-1+t_{n-1})) = S_1 + \frac{t_{n-1}(1+S_1)}{n-1} < S_1.$$

Suppose that $S_i < S_{i-1} < -1$ for some i. Then, form (2), we have

$$\begin{aligned} S_{i+1} &= -\frac{1}{n-i}(S_i s_{n-i} + S_{i-1}t_{n-i}) = \frac{1}{n-i}(S_i(-s_{n-i}) - S_{i-1}t_{n-i}) \\ &< \frac{1}{n-i}(S_i(n-i+t_{n-i}) - S_{i-1}t_{n-i}) \\ &= S_i + t_{n-i}(\frac{S_i - S_{i-1}}{n-i}). \end{aligned}$$

Since $S_i < S_{i-1}$, we have $S_{i+1} < S_i$. By the mathematical induction, we have

$$S_{i+1} < S_i$$

for all $i = 1, 2, \cdots, n$. Hence, for a basis $(u_0, u_1, \cdots, u_n)^T$ of $\mathcal{N}(L_{sub})$, we have

$$u_0 > u_1 > \cdots > u_n = 1.$$

Since $\beta > 0$, we have $\det(\mathcal{L}_\beta) > 0$. Further, since cofactors of diagonal entries of $\mathcal{L}_\beta$ are positive values, we have $r_0 > 0$. Therefore, we have $r_0 > r_1 > \cdots > r_{n-1} > r_n > 0$. □

Example 3. Let L_{sub} be a 5×6 matrix obtained by Step 1,2,3 and 4 in Section 3. Then, L_{sub} is given by

$$\begin{pmatrix} 1 & 1-10(\beta+1) & 8 & 0 & 0 & 0 \\ 0 & 2 & 2-10(\beta+1) & 6 & 0 & 0 \\ 0 & 0 & 3 & 3-10(\beta+1) & 4 & 0 \\ 0 & 0 & 0 & 4 & 4-10(\beta+1) & 2 \\ 0 & 0 & 0 & 0 & 5 & 5-10(\beta+1) \end{pmatrix}.$$

Choosing $\beta = \frac{1}{10}$, then we obtain a basis of $\mathcal{N}(L_{sub})$ as

$$\left(\frac{152}{3}, \frac{36}{5}, \frac{8}{3}, \frac{8}{5}, \frac{6}{5}, 1\right).$$

Also, $\overline{L}_{sub}$ is

$$\begin{pmatrix} -11 & 10 & 0 & 0 & 0 & 0 \\ 1 & -10 & 8 & 0 & 0 & 0 \\ 0 & 2 & -9 & 6 & 0 & 0 \\ 0 & 0 & 3 & -8 & 4 & 0 \\ 0 & 0 & 0 & 4 & -7 & 2 \\ 0 & 0 & 0 & 0 & 5 & -6 \end{pmatrix}$$

and $\det(\overline{L}_{sub}) = 58240$, $c = 1200/58240 = 15/728$. Thus,

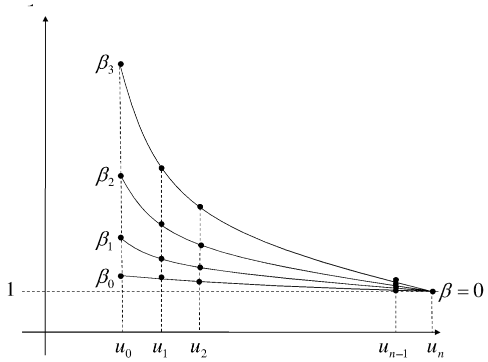
$$(r_0, r_1, r_2, r_3, r_4, r_5) = \frac{15}{728}\left(\frac{152}{3}, \frac{36}{5}, \frac{8}{3}, \frac{8}{5}, \frac{6}{5}, 1\right).$$

Note. (1) Let $(u_0, u_1, \cdots, u_n)$ be a basis of $\mathcal{N}(L_{sub})$ with $u_n = 1$. For all i with $i = 0, 1, \cdots, n-1$, if $\beta > 0$ increases, u_i increases. Furthermore, u_i approaches 1^+ as β approaches 0^+.

(2) [5] Let L_0 be a $n \times n$ matrix obtained by the removal of the first column of L_{sub}. Let L_i be a $(n-i) \times (n-i)$ matrix obtained by the removal from the first row(respectively, column) to the i-th row(respectively, column) of L_0 and let $(u_0, u_1, \cdots, u_n)$ be a basis of $\mathcal{N}(L_{sub})$ with $u_n = 1$. Then we have

$$u_i = (-1)^{n-i}\frac{i!\ \det(L_i)}{n!}, \ (i = 0, 1, \cdots, n-1),$$

where $\det(L_n)$



5. A coset weight enumerator as a Green's function $\mathcal{G}_\beta$

In this section, we investigate the relationship between an a coset weight enumerator of a linear code $\mathcal{C}$ and a Green's function $\mathcal{G}_\beta$. We begin, we introduce some definitions.

Definition 4. *(1) Let $\mathcal{C}$ be a subset of $\mathbb{F}_q$ of length n. We arrange elements of $\mathcal{C}$ according to their Hamming weight. Define*

$$\mathcal{P}_i = \begin{cases} 1, & \text{if } i\text{-th element is in } \mathcal{C}, \\ 0, & \text{if } i\text{-th element is not in } \mathcal{C}, \end{cases} .$$

This $q^n \times 1$ vector $\mathcal{P} = pos(\mathcal{C}) = (\mathcal{P}_i)_{1\le i\le q^n}$ is called a position vector of $\mathcal{C}$.

(2) For a subset $\mathcal{C}$ of $\mathbb{F}_q$ of length n and $y_i \in \mathcal{C}^{-1} = \mathbb{F}_q^n - \mathcal{C}$, let

$$S(y_i) = \{z \mid W(\mathcal{C}+z) = W(\mathcal{C}+y_i), z \in \mathcal{C}^{-1}\}$$

and t the number of the distinct set $S(y_i)$'s. Then, $\mathcal{P}^{(i)} = pos_{(i)}(\mathcal{C}^{-1})$ is the position vector of the set $S(y_i)$ for $i = 1, 2, \cdots, t$, and $\mathcal{P}^{(i)}$ is called the coset position vector of $\mathcal{C}^{-1}$ with respect to y_i.

(3) Let $\mathcal{G}_\beta$ be a Green's function as in Theorem 2 (1). That is, $\mathcal{G}_\beta = r_0A_0 + r_1A_1 + \cdots + r_nA_n$. For $z \in S(y_i)$, let $W(\mathcal{C}+z) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$, and let a value $\alpha^{(i)}$ denote $c_0r_0 + c_1r_1 + \cdots + c_nr_n$, then $\alpha^{(i)}$ is called the coset weight enumerator value of $S(y_i)$ $(i = 1, 2, \cdots, t)$. Also, we define that $\alpha^{(0)}$ is a coset weight enumerator value of $\mathcal{C}$. That is,

$$\begin{cases} \alpha^{(0)}, & z \in \mathcal{C} \\ \alpha^{(i)}, & z \in S(y_i). \end{cases}$$

Example 5. We arrange all elements of $\mathbb{F}_2^3$ as 000, 100, 010, 001, 110, 101, 011, 111. Let $\mathcal{C} = \{000, 100\}$. Then the position vector of $\mathcal{C}$ is $\mathcal{P}=pos(\mathcal{C}) = (1,1,0,0,0,0,0,0)^T$. We consider a coset $\mathcal{C}+y$ for $y \in \mathcal{C}^{-1}$. Then $\mathcal{C}+y$ are $\{010,\ 110\}, \{001,\ 101\}$, and $\{011,\ 111\}$. Thus $\mathcal{P}^{(1)} = (0,0,1,1,1,1,0,0)$, $\mathcal{P}^{(2)} = (0,0,0,0,0,0,1,1)$. Moreover, $\alpha^{(0)} = r_0+r_1$, $\alpha^{(1)} = r_1+r_2$ and $\alpha^{(2)} = r_2+r_3$. Now, we consider a Green's function $\mathcal{G}_\beta = r_0A_0+r_1A_1+r_2A_2+r_3A_3$ over $\mathbb{F}_2^3$ as follows :

$$\begin{pmatrix} r_0 & r_1 & r_1 & r_1 & r_2 & r_2 & r_2 & r_3 \\ r_1 & r_0 & r_2 & r_2 & r_1 & r_1 & r_3 & r_2 \\ r_1 & r_2 & r_0 & r_2 & r_1 & r_3 & r_1 & r_2 \\ r_1 & r_2 & r_2 & r_0 & r_3 & r_1 & r_1 & r_2 \\ r_2 & r_1 & r_1 & r_3 & r_0 & r_2 & r_2 & r_1 \\ r_2 & r_1 & r_3 & r_1 & r_2 & r_0 & r_2 & r_1 \\ r_2 & r_3 & r_1 & r_1 & r_2 & r_2 & r_0 & r_1 \\ r_3 & r_2 & r_2 & r_2 & r_1 & r_1 & r_1 & r_0 \end{pmatrix}.$$

Then, $\mathcal{G}_\beta\mathcal{P}$ is $(\alpha^{(0)}, \alpha^{(0)}, \alpha^{(1)}, \alpha^{(1)}, \alpha^{(1)}, \alpha^{(1)}, \alpha^{(2)}, \alpha^{(2)})$. That is,

$$\mathcal{G}_\beta\mathcal{P} = \alpha^{(0)}\mathcal{P} + \alpha^{(1)}\mathcal{P}^{(1)} + \alpha^{(2)}\mathcal{P}^{(2)}.$$

In the following theorem, we show that for a given β and a given liner code $\mathcal{C}$ over $\mathbb{F}_q$ of length n, the coset weight enumerator values can be uniquely expressed as a linear combination of r_i's, where $\mathcal{G}_\beta = r_0A_0+r_1A_1+\cdots+r_nA_n$. Furthermore, we find the exact number of distinct coset weight enumerators of $\mathcal{C}$ by expressing $\mathcal{G}_\beta\mathcal{P}$ as a linear combination of a position vector $\mathcal{P}$ of $\mathcal{C}$ and the coset position vectors $\mathcal{P}^{(i)}$ with coefficients $\alpha^{(i)}$.

Theorem 6. *Let $\mathcal{C}$ be a linear code over $\mathbb{F}_q$ of length n, and $\mathcal{P} = pos(\mathcal{C})$, $\mathcal{P}^{(i)} = pos_{(i)}(\mathcal{C}^{-1})$. Let $\mathcal{G}_\beta = r_0A_0 + r_1A_1 + \cdots + r_nA_n$ be a Green's function over $\mathbb{F}_q^n$ as in Theorem 2 and let $\alpha^{(i)}$ $(i = 0, 1, \cdots, t)$ be the coset weight enumerator values over $\mathbb{F}_q^n$. Then we have the following:*

(1) $\alpha^{(k)} = \alpha^{(l)}$ *if and only if* $c'_i = c''_i$ *for all* i, *where* $\alpha^{(k)} = \sum c'_i r_i$ *and* $\alpha^{(l)} = \sum c''_i r_i$.

(2) *There are* $t+1$ *distinict coset weight enumerators of* $\mathcal{C}$ *if and only if* $\mathcal{G}_\beta\mathcal{P} = \alpha^{(0)}\mathcal{P} + \alpha^{(1)}\mathcal{P}^{(1)} + \cdots + \alpha^{(t)}\mathcal{P}^{(t)}$ *for distinct positive* $\alpha^{(i)}$, *where* $\mathcal{P} + \mathcal{P}^{(1)} + \cdots + \mathcal{P}^{(t)} = \mathbf{1}$.

Proof. (1) ($\Rightarrow$) Suppose that $\alpha^{(k)} = \alpha^{(l)}$ for some k, l. Then, we have

$$\alpha^{(k)} = c'_0r_0 + c'_1r_1 + \cdots + c'_nr_n = c''_0r_0 + c''_1r_1 + \cdots + c''_nr_n = \alpha^{(l)}$$

for some c'_i's and c''_i's. This implies that

$$\begin{pmatrix} c'_0 & c'_1 & \cdots & c'_n \\ c''_0 & c''_1 & \cdots & c''_n \end{pmatrix} \begin{pmatrix} r_0 & 1 \\ r_1 & 1 \\ \vdots & \vdots \\ r_n & 1 \end{pmatrix} = \begin{pmatrix} \alpha^{(k)} & |C| \\ \alpha^{(k)} & |C| \end{pmatrix}.$$

Let $K = \begin{pmatrix} c'_0 & c'_1 & \cdots & c'_n \\ c''_0 & c''_1 & \cdots & c''_n \end{pmatrix}$, and we assume that rank$(K) = 2$. Then $w = (c'_0 - c''_0, c'_1 - c''_1, \cdots, c'_n - c''_n)$ is a non-zero vector, and w is orthogonal to both $(1, 1, \cdots, 1)$ and $(r_0, r_1, \cdots, r_n)$. Hence, w is contained in the nullspaces of $1 \times (n+1)$ matrices $(1\ 1\ \cdots\ 1)$ and $(r_0\ r_1\ \cdots\ r_n)$, respectively, where the nullspaces of $(1\ 1\ \cdots\ 1)$ and $(r_0\ r_1\ \cdots\ r_n)$ are same to the row spaces of

$$U = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \text{ and } V = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -r_0 \\ 0 & 1 & 0 & \cdots & 0 & -r_1 \\ 0 & 0 & 1 & \cdots & 0 & -r_2 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -r_{n-1} \end{pmatrix},$$

respectively. Let u_i and v_i be the i-th row vectors of U and V, respectively. Then, for $i = 1, 2, \cdots, n$, we have $u_i \notin span\{v_1, v_2, \cdots, v_n\}$, and u_i's are linearly independent. Suppose that $au_i + bu_j = t_1v_1 + \cdots + t_nv_n$ for some i, j, a and b. If all t_i's are 0, then we have $a = b = 0$ since u_i's are linearly independent. If $t_i \neq 0$ for some i, then neither a nor b is 0. Hence, there are j and k such that $b_k \neq 0$, $t_j \neq 0$, and

$$u_i = \sum_{b_k \neq 0} b_k u_k + \sum_{t_j \neq 0} t_j v_j.$$

Since $w = \sum g_i u_i$, we have

$$\begin{aligned} w &= \sum g_i \left(\sum_{b_k \neq 0} b_k u_k + \sum_{t_j \neq 0} t_j v_j \right) \\ &= \sum \sum_{b_k \neq 0} g_i b_k u_k + \sum \sum_{t_j \neq 0} g_i t_j v_j. \end{aligned}$$

Since $g_i b_k \neq 0$ and $g_i t_j \neq 0$ for some i, j, k, we have $w \notin span\{v_1, \cdots, v_n\}$. Hence, in order that w is orthogonal to both $(1, 1, \cdots, 1)$ and $(r_0, r_1, \cdots, r_n)$, it should hold that $w = O$. Thus, we have $c'_0 = c''_0$, $c'_1 = c''_1$, $\cdots$, $c'_n = c''_n$, i.e., we have $\text{rank}(K) = 1$, which is a contradiction.
($\Leftarrow$) It is obvious.
(2) ($\Rightarrow$) Suppose that there are $t + 1$ distinct coset weight enumerators on $\mathcal{C}$. Then by (1), there exist $t + 1$ distinct values $\alpha^{(i)}$ $(i = 0, 1, \cdots, t)$. Since $\mathcal{G}_\beta = r_0 A_0 + r_1 A_1 + \cdots + r_n A_n$, by definition 4, $\mathcal{G}_\beta \mathcal{P}$ is a linear combination of $\mathcal{P}$ and $\mathcal{P}^{(i)}$ $(0 \leq i \leq t)$ with coefficients $\alpha^{(i)}$. Also, It is clear, $\mathcal{P} + \mathcal{P}^{(1)} + \mathcal{P}^{(2)} + \cdots + \mathcal{P}^{(t)} = \mathbf{1}$.
($\Leftarrow$) By (1) and definition 4, it is obvious. □

Example 7. Let $\mathcal{C}$ be a linear code over $\mathbb{F}_2$ of length 4 as follows :

$$\{0000,\ 1010,\ 1101,\ 0111\}.$$

Then, a position vector $\mathcal{P}$ of $\mathcal{C}$ is as follows :

$$\mathcal{P} = (1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0)^T$$

Let $\mathcal{G}_\beta$ be a Green's function over $\mathbb{F}_2^4$ with $\beta = \frac{1}{4}$. Then $\mathcal{G}_\beta$ is as follows :

$$\mathcal{G}_\beta = \frac{332}{315} A_0 + \frac{20}{63} A_1 + \frac{8}{45} A_2 + \frac{8}{63} A_3 + \frac{32}{315} A_4,$$

where A_i $(i = 0, 1, \cdots, 4)$ are the adjacency matrices of $H(4, 2)$. That is,

$$r_0 = \frac{332}{315}, r_1 = \frac{20}{63}, r_2 = \frac{8}{45}, r_3 = \frac{8}{63}, r_4 = \frac{32}{315}.$$

Thus, we have

$$\begin{aligned} \mathcal{G}_\beta \mathcal{P} &= (\frac{52}{35}, \frac{32}{35}, \frac{4}{5}, \frac{32}{35}, \frac{4}{5}, \frac{4}{5}, \frac{52}{35}, \frac{4}{5}, \frac{4}{5}, \frac{32}{35}, \frac{4}{5}, \frac{4}{5}, \frac{52}{35}, \frac{4}{5}, \frac{52}{35}, \frac{32}{35}) \\ &= \frac{52}{35} \mathcal{P} + \frac{32}{35} \mathcal{P}^{(1)} + \frac{4}{5} \mathcal{P}^{(2)}. \end{aligned}$$

Therefore, by Theorem 6, there are exactly three distinct coset weight enumerators of $\mathcal{C}$. In fact, the cosets of $\mathcal{C}$ are

$$\left\{ \begin{array}{c} 0000 \\ 1010 \\ 1101 \\ 0111 \end{array} \right\}, \left\{ \begin{array}{c} 1000 \\ 0010 \\ 0101 \\ 1111 \end{array} \right\}, \left\{ \begin{array}{cc} 0100 & 0001 \\ 1110 & 1011 \\ 1001 & 1100 \\ 0011 & 0110 \end{array} \right\},$$

and the coset Hamming enumerators are $1+x^2+2x^3$, $2x+x^2+x^4$, $x+2x^2+x^3$. Therefore $\mathcal{G}_\beta \mathcal{P}$ is $(r_0 + r_2 + 2r_3)\mathcal{P} + (2r_1 + r_2 + r_4)\mathcal{P}^{(1)} + (r_1 + 2r_2 + r_3)\mathcal{P}^{(2)}$.

For two linear codes over $\mathbb{F}_q$ of length n with the same dimension, the following result shows the relation between their coset weight enumerator values $\alpha^{(i)}$ and their number of the distinct coset weight enumerators when $\mathcal{G}_\beta\mathcal{P}$ and $\mathcal{G}_\beta\mathcal{P}'$ are expressed as a linear combination of coset position vectors $\mathcal{P}^{(i)}$ with coefficients $\alpha^{(i)}$.

Theorem 8. *Let $\mathcal{C}$ and $\mathcal{C}'$ be linear codes over $\mathbb{F}_q$ of length n with $\dim(\mathcal{C}) = \dim(\mathcal{C}')$, and let $\mathcal{P}$ (resp. $\mathcal{P}^{(i)}$) and $\mathcal{P}'$ (resp. $\mathcal{P}'^{(i)}$) be the position vectors (resp. coset position vectors) of $\mathcal{C}$ and $\mathcal{C}'$, respectively. For a Green's function $\mathcal{G}_\beta$ over $\mathbb{F}_q^n$, suppose that*

$$\mathcal{G}_\beta\mathcal{P} = \alpha^{(0)}\mathcal{P} + \alpha^{(1)}\mathcal{P}^{(1)} + \cdots + \alpha^{(t)}\mathcal{P}^{(t)},$$
$$\mathcal{G}_\beta\mathcal{P}' = \alpha'^{(0)}\mathcal{P}' + \alpha'^{(1)}\mathcal{P}'^{(1)} + \cdots + \alpha'^{(l)}\mathcal{P}'^{(l)}.$$

Then, we have as follows :

(1) $\sum \alpha^{(j)} \geq \sum \alpha'^{(j)} \;\Rightarrow\; t \geq l$,

(2) $t < l \;\Rightarrow\; \sum \alpha^{(j)} < \sum \alpha'^{(j)}$,

(3) $\sum \alpha^{(j)} = \sum \alpha'^{(j)} \;\Leftrightarrow\; \sum \alpha^{(j)}$ *and* $\sum \alpha'^{(j)}$ *have the same linear combination of* $r_0, r_1, \cdots, r_n$.

Proof. (1) Suppose that $\sum \alpha^{(j)} \geq \sum \alpha'^{(j)}$. Then for $\beta > 0$,

$$\sum \alpha^{(j)} = d_0r_0 + d_1r_1 + \cdots + d_nr_n \geq d_0'r_0 + d_1'r_1 + \cdots + d_n'r_n = \sum \alpha'^{(j)}.$$

Let $(u_0\ u_1\ \cdots\ u_n)$ be a basis of L_{sub} with $u_n = 1$. Then $(u_0\ u_1\ \cdots\ u_n) \to (1, 1, \cdots, 1)^+$ as $\beta \to 0^+$. Therefore

$$\begin{aligned}\lim_{\beta\to 0^+} &(d_0r_0 + d_1r_1 + \cdots + d_nr_n \geq d_0'r_0 + d_1'r_1 + \cdots + d_n'r_n)\\ &= (d_0 + d_1 + \cdots + d_n \geq d_0' + d_1' + \cdots + d_n').\end{aligned}$$

Since $\sum d_i = t|\mathcal{C}|$ and $\sum d_i' = l|\mathcal{C}|$, $t|\mathcal{C}| \geq l|\mathcal{C}| \;\Leftrightarrow\; t \geq l$.
(2) by (1), it is obvious.
(3) ($\Rightarrow$) Let $\sum \alpha^{(j)} = \sum_{i=0}^n d_ir_i$ and $\sum \alpha'^{(j)} = \sum_{i=0}^n d_i'r_i$. Suppose that $\sum \alpha^{(j)} = \sum \alpha'^{(j)}$ for any $\beta > 0$. Then $(d_0 - d_0', \cdots, d_n - d_n')$ is an orthogonal with $(r_0, r_1, \cdots, r_n)$. Let $\beta_i > 0$ $(i = 0, 1, \cdots, n)$ with $0 < \beta_0 < \beta_1 < \cdots < \beta_n$ and let $(u_0^{(i)}, u_1^{(i)}, \cdots, u_n^{(i)})$ be a basis of $\mathcal{N}(L_{sub})$ with respect to $\beta_i > 0$. Define a $(n+1) \times (n+1)$ matrix $B = (u_j^{(i)})$ as follows :

$$\begin{pmatrix} u_0^{(0)} & u_1^{(0)} & \cdots & \cdots & u_n^{(0)} \\ u_0^{(1)} & u_1^{(1)} & \cdots & \cdots & u_n^{(1)} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ u_0^{(n)} & u_1^{(n)} & \cdots & \cdots & u_n^{(n)} \end{pmatrix},$$

where $u_n^{(i)} = 1$ $(i = 0, 1, \cdots, n)$. Since $u_0^{(i)} > u_1^{(i)} > \cdots > u_n^{(i)} = 1$ and $u_j^{(0)} < u_j^{(1)} < \cdots < u_j^{(n)}$ $(i, j = 0, 1, \cdots, n,\ j \neq n)$, B is an invertible matrix. That is, $\mathrm{rank}(B) = n + 1$. Since a nullspace of $(d_0 - d_0', \cdots, d_n -$

d'_n) is a n-dimensional space, $(d_0 - d'_0, \cdots, d_n - d'_n)$ is not orthogonal with $(r_0, r_1, \cdots, r_n)$ for some $\beta > 0$. Therefore $(d_0 - d'_0, \cdots, d_n - d'_n) = O$, that is, $\sum \alpha^{(j)}$ and $\sum \alpha'^{(j)}$ have the same linear combination of $r_0, r_1, \cdots, r_n$.
($\Leftarrow$) It is obvious. □

Example 9. Let $\mathcal{C}_i$ $(i = 1, 2)$ be a linear codes over $\mathbb{F}_2$ of length 5 as follows :

$$\mathcal{C}_1 = \{00000,\ 10000\},\ \mathcal{C}_2 = \{00000,\ 01111\}.$$

And let $\mathcal{P}$ (resp. $\mathcal{P}^{(i)}$) and $\mathcal{P}'$ (resp. $\mathcal{P}'^{(i)}$) be the position vectors (resp. coset position vectors) of $\mathcal{C}_1$, $\mathcal{C}_2$, respectively. Then a coset enumerators of $\mathcal{C}_1$ and $\mathcal{C}_2$ are

$$\begin{aligned}&1+x,\ x+x^2,\ x^2+x^3,\ x^3+x^4,\ x^4+x^5\\ &1+x^4,\ x+x^5,\ x+x^3,\ x^2+x^4,\ 2x^2,\ 2x^3\end{aligned}$$

respectively. Thus we have

$$\begin{aligned}\mathcal{G}_\beta\mathcal{P} &= (r_0+r_1)\mathcal{P} + (r_1+r_2)\mathcal{P}^{(1)} + (r_2+r_3)\mathcal{P}^{(2)}\\ &\qquad + (r_3+r_4)\mathcal{P}^{(3)} + (r_4+r_5)\mathcal{P}^{(4)},\\ \mathcal{G}_\beta\mathcal{P}' &= (r_0+r_4)\mathcal{P}' + (r_1+r_5)\mathcal{P}'^{(1)} + (r_1+r_3)\mathcal{P}'^{(2)}\\ &\qquad + (r_2+r_4)\mathcal{P}'^{(3)} + (2r_2)\mathcal{P}'^{(4)} + (2r_3)\mathcal{P}'^{(5)}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\sum \alpha^{(j)} &= r_0 + 2r_1 + 2r_2 + 2r_3 + 2r_4 + r_5,\\ \sum \alpha'^{(j)} &= r_0 + 2r_1 + 3r_2 + 3r_3 + 2r_4 + r_5.\end{aligned}$$

Since $r_0 > r_1 > r_2 > \cdots > r_n > 0$, $\sum \alpha^{(j)} < \sum \alpha'^{(j)}$. And the numbers of distinct coset weight enumerators of $\mathcal{C}$ and $\mathcal{C}'$ are 5 and 6 respectively.

Example 9 shows that if $\sum \alpha^{(j)} > \sum \alpha'^{(j)}$, then $t > l$. However, it is not true in general. Example 10 shows this case that $t = l$ but $\sum \alpha^{(j)} \neq \sum \alpha'^{(j)}$.

Example 10. Let $\mathcal{C}_i$ $(i = 1, 2, 3)$ be a linear codes over $\mathbb{F}_2$ of length 5 as follows :

$$\begin{aligned}\mathcal{C}_1 &= \{00000,\ 10110,\ 11101,\ 01011\},\\ \mathcal{C}_2 &= \{00000,\ 10000,\ 01000,\ 11000\},\\ \mathcal{C}_3 &= \{00000,\ 11000,\ 00111,\ 11111\},\end{aligned}$$

And let $\mathcal{P}$,$\mathcal{P}'$ and $\mathcal{P}''$ (resp. $\mathcal{P}^{(i)}, \mathcal{P}'^{(i)}$ and $\mathcal{P}''^{(i)}$) be the position vectors (resp. coset position vectors) of $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, respectively. Then a coset enumerators of $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are

$$\begin{aligned}&1+2x^3+x^4,\ x+x^2+x^3+x^4,\ x+2x^2+x^5,\ 2x^2+x^3,\\ &1+2x^1+x^2,\ x+2x^2+x^3,\ x^2+2x^3+x^4,\ x^3+2x^4+x^5,\\ &1+x^2+x^3+x^5,\ 2x+2x^4,\ x+x^2+x^3+x^4,\ 2x^2+2x^3,\end{aligned}$$

respectively. Moreover, $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ has the same number of distinct coset weight enumerators. Thus, we have

$$\begin{aligned}\mathcal{G}_\beta\mathcal{P} =&(r_0+2r_3+r_4)\mathcal{P}+(r_1+r_2+r_3+r_4)\mathcal{P}^{(1)}\\&+(r_1+2r_2+r_5)\mathcal{P}^{(2)}+(2r_2+2r_3)\mathcal{P}^{(3)},\\ \mathcal{G}_\beta\mathcal{P}' =&(r_0+2r_1+r_2)\mathcal{P}'+(r_1+2r_2+r_3)\mathcal{P}'^{(1)}\\&+(r_2+2r_3+r_4)\mathcal{P}'^{(2)}+(r_3+2r_4+r_5)\mathcal{P}''^{(3)},\\ \mathcal{G}_\beta\mathcal{P}'' =&(r_0+r_2+r_3+r_5)\mathcal{P}''+(2r_1+2r_4)\mathcal{P}''^{(1)}\\&+(r_1+r_2+r_3+r_4)\mathcal{P}''^{(2)}+(2r_2+2r_3)\mathcal{P}''^{(3)}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\sum\alpha^{(j)} =&r_0+2r_1+5r_2+5r_3+2r_4+r_5,\\ \sum\alpha'^{(j)} =&r_0+3r_1+4r_2+4r_3+3r_4+r_5,\\ \sum\alpha''^{(j)} =&r_0+3r_1+4r_2+4r_3+3r_4+r_5.\end{aligned}$$

Choosing $\beta=\frac{1}{5}$, we have $(r_0,r_1,r_2,r_3,r_4,r_5)=\frac{1}{4563}(730,183,92,62,48,40)$, and $\sum\alpha^{(j)}=\frac{2002}{4563}$ and $\sum\alpha'^{(j)}=\sum\alpha''^{(j)}=\frac{2079}{4563}$.

6. Application : Explicit lower bounds of the Cheeger constant of a graph with respect to Hamming distance 1

Let Γ_1 be a graph with respect to a Hamming distance 1 over $\mathbb{F}_q^n$. Then Γ_1 is a distance regular graph. The concept of the Cheeger ratio and the Cheeger constant of subset S of set of vertices of a graph G was introduced in [2]. Let Γ_1 be a graph as G, and let $\mathcal{C}$ be a subset of $\mathbb{F}_q$ of length n as a set S. Then, in this section, we obtain the lower bounds of the Cheeger ratio and the Cheeger constant of $\mathcal{C}$ on Γ_1. Now, we introduce some definitions in graph theory.

Definition 11. *Let S be a subset of set of vertices in a graph $G=(V,E)$, where V is a set of vertices of G and E is a set of edges of G.*
(1) The edge boundary of S, denoted by $\partial(S)$ is defined as follows :

$$\partial(S)=\{\{u,v\}\in E(G)\mid u\in S \text{ and } v\in V-S\},$$

where $E(G)$ is a edge set of G.
(2) If $S\neq\varnothing$, then the volume of S, denoted by vol(S) *is defined as follows :*

$$\mathrm{vol}(S)=\sum_{x\in S}k_x,$$

where k_x is a valency of x in G. The volume of G is demoted by

$$\mathrm{vol}(G)=\sum_{x}d_x.$$

(3) The Cheeger ratio of S*, denoted by* $h(S)$ *is defined as*

$$h(S) = \frac{|\partial(S)|}{\min\{\mathrm{vol}(S),\ \mathrm{vol}(G) - \mathrm{vol}(S)\}}.$$

(4) The Cheeger constant of G*, denoted by* h_G *is defined as*

$$h_G = \min_S h(S).$$

We introduce the definition of a *personalized PageRank* $pr(\alpha, s)$ and several facts that relate the Cheeger ratio and constant to PageRank [2].

Theorem 12. [F.Chung, 2] *Let* S *be a subset of set of vertices in a graph* G*. Let* s *be a seed vector, and let* α *be the jumping constant. Then the PageRank* $pr(\alpha, s)$ *is defined as*

$$pr(\alpha, s) = \alpha \sum_{k=0}^{\infty} (1-\alpha)^k s W^k,$$

where W *denotes a lazy walk of* G*, defined by* $W = \frac{I+P}{2}$*.*

(a) Then we have

$$pr(\alpha, s) = \beta s \mathcal{G}_\beta,$$

where $\beta = \frac{2\alpha}{1-\alpha}$*. Also, for a subset* S *of set of vertices in a graph* G*, we have*

$$pr(\alpha, s)(S) = \beta s \mathcal{G}_\beta \chi_S,$$

where χ_S *is the characteristic function of* S*.*

(b) For a subset S*, the probability function* f_C *satisfies*

$$pr(\alpha, f_C)(S) \geq 1 - \frac{1-\alpha}{2\alpha} h(S).$$

(c) Let T *be a subset of* S *with* $\mathrm{vol}(T) \geq \mathrm{vol}(S)/2$ *such that for any* $u \in T$*, the personalized pagerank* $pr(\alpha, u)$ *satisfies*

$$pr(\alpha, u)(S) \geq 1 - \frac{1-\alpha}{\alpha} h_S.$$

Let $\mathcal{G}_\beta$ be a Green's function over $\mathbb{F}_q^n$. In Theorem 2, $\mathcal{G}_\beta$ is expressed by

$$\mathcal{G}_\beta = r_0 A_0 + r_1 A_1 + \cdots + r_n A_n$$

with $r_0 > r_1 > \cdots > r_n$. In Theorem 6, $\mathcal{G}_\beta \mathcal{P}$ is expressed by

$$\mathcal{G}_\beta \mathcal{P} = \alpha^{(0)} \mathcal{P} + \alpha^{(1)} \mathcal{P}^{(1)} + \cdots + \alpha^{(t)} \mathcal{P}^{(t)}$$

for some t, where $\mathcal{P}$ is a position vector of a linear code $\mathcal{C}$ and $\mathcal{P}^{(i)}$ $(i = 1, 2, \ldots, t)$ are the coset position vectors.

The following theorem shows that the lower bounds of the Cheeger ratio and the Cheeger constant of $\mathcal{C}$ on Γ_1 can be explicitly determined by β, $\alpha^{(0)}$ and r_0, which are the values coming from $\mathcal{G}_\beta$ and a coset weight enumerator value of $\mathcal{C}$.

Theorem 13. *Let Γ_1 be a distance regular graph with respect to a Hamming distance 1 over $\mathbb{F}_q^n$, and $\mathcal{C}$ be a subset of set of vertices of Γ_1. Let $\beta = \frac{2\alpha}{1-\alpha}$ for a jumping constant α, $\alpha^{(0)}$ be given as in definition 4(3), and r_0 be given as in Theorem 2. Then we have the following.*

(1) The probability function $f_{\mathcal{C}}$ and the Cheeger ratio $h(\mathcal{C})$ satisfies the following:

$$If\ \mathcal{C}\ is\ a\ linear\ code, then \begin{cases} pr(\alpha, f_{\mathcal{C}})(\mathcal{C}) = \beta\alpha^{(0)} \\ h(\mathcal{C}) \geq \beta(1-\beta\alpha^{(0)}), \end{cases}$$

$$If\ \mathcal{C}\ is\ a\ nonlinear\ code, then \begin{cases} pr(\alpha, f_{\mathcal{C}})(\mathcal{C}) \leq \beta r_0|\mathcal{C}| \\ h(\mathcal{C}) \geq \beta(1-\beta r_0|\mathcal{C}|). \end{cases}$$

(2) Let T be a subset of $\mathcal{C}$ over $\mathbb{F}_q$ of length n with $\mathrm{vol}(T) \geq \mathrm{vol}(\mathcal{C})/2$, then for any $u \in T$, $pr(\alpha, u)(\mathcal{C})$ and the Cheeger constant $h_{\mathcal{C}}$ satisfies the following:

$$If\ \mathcal{C}\ is\ a\ linear\ code, then \begin{cases} pr(\alpha, u)(\mathcal{C}) = \beta\alpha^{(0)} \\ h_{\mathcal{C}} \geq \frac{1}{2}\beta(1-\beta\alpha^{(0)}), \end{cases}$$

$$If\ \mathcal{C}\ is\ a\ nonlinear\ code, then \begin{cases} pr(\alpha, u)(\mathcal{C}) \leq \beta r_0|\mathcal{C}| \\ h_{\mathcal{C}} \geq \frac{1}{2}\beta(1-\beta r_0|\mathcal{C}|). \end{cases}$$

Proof. Let $\mathcal{P}$ be a position vector of $\mathcal{C}$, and let $\mathcal{G}_\beta$ be a Green's function as in Theorem 6. Let $\mathcal{C}$ be a subset over $\mathbb{F}_q$ of length n, and we consider the probability distribution which is

$$f_{\mathcal{C}}(x) = \begin{cases} \frac{k_x}{\mathrm{vol}(\mathcal{C})}, & x \in \mathcal{C}, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can write $f_{\mathcal{C}} = \frac{1}{\mathrm{vol}(\mathcal{C})}\chi_{\mathcal{C}} D$, where $\chi_{\mathcal{C}}$ is the characteristic fuction of $\mathcal{C}$, and D is the diagonal matrix with $D(x,x) = k_x$. Since k_x is $n(q-1)$ on Γ_1, we have

$$\mathrm{vol}(\mathcal{C}) = \sum_{x \in \mathcal{C}} k_x = n(q-1)|\mathcal{C}|,$$

and $\chi_{\mathcal{C}}$ is a position vector of $\mathcal{C}$, that is, $\chi_{\mathcal{C}} = poc(\mathcal{C}) = \mathcal{P}$. Thus,

$$f_{\mathcal{C}} = \frac{1}{\mathrm{vol}(\mathcal{C})}\chi_{\mathcal{C}} D = \frac{1}{n(q-1)|\mathcal{C}|}\mathcal{P}D = \frac{1}{n(q-1)|\mathcal{C}|}n(q-1)\mathcal{P} = \frac{1}{|\mathcal{C}|}\mathcal{P}.$$

(1) [Case I : $\mathcal{C}$ is a linear code.] Since $pr(\alpha, f_{\mathcal{C}})(\mathcal{C}) = \beta f_{\mathcal{C}} \mathcal{G}_\beta \mathcal{P} = \frac{\beta}{|\mathcal{C}|}\mathcal{P}^T \mathcal{G}_\beta \mathcal{P}$ by Theorem 12 (a). Since $\mathcal{C}$ is a linear, by Theorem 6, the coefficient of $\mathcal{P}$ on $\mathcal{G}_\beta \mathcal{P}$ is $\alpha^{(0)}$. Thus we have

$$pr(\alpha, f_{\mathcal{C}})(\mathcal{C}) = \frac{\beta}{|\mathcal{C}|}\alpha^{(0)}|\mathcal{C}| = \beta\alpha^{(0)} = \frac{2\alpha}{1-\alpha}\alpha^{(0)}.$$

Therefore, by Theorem 12 (b), we have

$$\begin{aligned} &\frac{2\alpha}{1-\alpha}\alpha^{(0)} \geq 1 - \frac{1-\alpha}{2\alpha}h(\mathcal{C}) \\ \Leftrightarrow\ & h(\mathcal{C}) \geq \frac{2\alpha}{1-\alpha}\Big(1 - \frac{2\alpha}{1-\alpha}\alpha^{(0)}\Big) \\ \Leftrightarrow\ & h(\mathcal{C}) \geq \beta(1-\beta\alpha^{(0)}). \end{aligned}$$

[Case II : $\mathcal{C}$ is a nonlinear code.] Since $pr(\alpha, f_{\mathcal{C}})(\mathcal{C}) = \beta f_{\mathcal{C}} \mathcal{G}_\beta \mathcal{P} = \frac{\beta}{|\mathcal{C}|}\mathcal{P}^T \mathcal{G}_\beta \mathcal{P}$ by Theorem 12 (a). Since $\mathcal{C}$ is a nonlinear and $r_0 > r_1 > \cdots > r_n$ by Theorem 2 (iii). We have

$$pr(\alpha, f_{\mathcal{C}})(\mathcal{C}) = \frac{\beta}{|\mathcal{C}|}\mathcal{P}^T \mathcal{G}_\beta \mathcal{P} \le \frac{\beta}{|\mathcal{C}|} r_0 |\mathcal{C}|^2 = \beta r_0 |\mathcal{C}| = \frac{2\alpha}{1-\alpha} r_0 |\mathcal{C}|.$$

Therefore, by Theorem 12 (b), we have

$$\begin{aligned} &\frac{2\alpha}{1-\alpha} r_0 |\mathcal{C}| \ge 1 - \frac{1-\alpha}{2\alpha} h(\mathcal{C}) \\ \Leftrightarrow\ & h(\mathcal{C}) \ge \frac{2\alpha}{1-\alpha}\Big(1 - \frac{2\alpha}{1-\alpha} r_0 |\mathcal{C}|\Big) \\ \Leftrightarrow\ & h(\mathcal{C}) \ge \beta(1 - \beta r_0 |\mathcal{C}|). \end{aligned}$$

(2) [Case I : $\mathcal{C}$ is a linear code.] Since $u \in T \subset \mathcal{C}$, we have $pr(\alpha, u)(\mathcal{C}) = \beta \chi_u \mathcal{G}_\beta \mathcal{P} = \beta \alpha^{(0)}$. Thus, by Theorem 12 (c), we have

$$\begin{aligned} & pr(\alpha, u)(\mathcal{C}) \ge 1 - \frac{1-\alpha}{\alpha} h_{\mathcal{C}} \\ \Leftrightarrow\ & \beta\alpha^{(0)} \ge 1 - \frac{1-\alpha}{\alpha} h_{\mathcal{C}} \\ \Leftrightarrow\ & h_{\mathcal{C}} \ge \frac{\alpha}{1-\alpha}(1 - \beta\alpha^{(0)}) \\ \Leftrightarrow\ & h_{\mathcal{C}} \ge \frac{1}{2}\beta(1 - \beta\alpha^{(0)}). \end{aligned}$$

[Case II : $\mathcal{C}$ is a nonlinear code.] Since, $r_0 > r_1 > \cdots > r_n$ by Theorem 2 (iii), we have $pr(\alpha, u)(\mathcal{C}) = \beta \chi_u \mathcal{G}_\beta \mathcal{P} \le \beta r_0 |\mathcal{C}|$. Thus, by Theorem 12 (c), we have

$$\begin{aligned} & pr(\alpha, u)(\mathcal{C}) \ge 1 - \frac{1-\alpha}{\alpha} h_{\mathcal{C}} \\ \Leftrightarrow\ & \beta r_0 |\mathcal{C}| \ge 1 - \frac{1-\alpha}{\alpha} h_{\mathcal{C}} \\ \Leftrightarrow\ & h_{\mathcal{C}} \ge \frac{\alpha}{1-\alpha}(1 - \beta r_0 |\mathcal{C}|) \\ \Leftrightarrow\ & h_{\mathcal{C}} \ge \frac{1}{2}\beta(1 - \beta r_0 |\mathcal{C}|). \end{aligned}$$

□

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